

## Singular and Ill-conditioned Linear Systems

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Solving systems of linear equations in the form of  $Ax = b$  is a fundamental problem in applied mathematics. A linear system is singular when the matrix  $A$  is rank-deficient and there are either infinitely many solutions or no solutions at all. Solving such a problem is a so-called ill-posed problem. In practical applications, singular linear systems are presented with empirical data and the matrices are highly ill-conditioned. As a result, solving such linear systems remains a formidable challenge. The elementary methods in linear algebra textbooks are available but not practical for numerical computation. In general, conventional wisdom suggests that singular linear systems should be avoided. As a result, the topic of solving singular linear systems is almost never mentioned in numerical analysis. In this talk, we shall demonstrate the usual pitfalls of standard methods for solving singular linear systems and why they fail. It is well known that singular and highly ill-conditioned linear systems are hypersensitive to data perturbations and round off errors. A small perturbation can result in huge errors in the computed solution. On the other hand, the hypersensitivity is one directional: A tiny perturbation in the matrix entries can only increase the rank and never decrease it. In other words, tiny perturbations can only decrease the singularity. Furthermore, when the singularity is maintained, the solutions are Lipschitz continuous with respect to data. In geometric terms, linear systems of the same singularity form complex analytic manifolds of a positive codimension and embedded in similar manifolds of lower codimensions. From this observation, we can formulate the notion of the numerical general solution to linear systems within an error tolerance. The numerical general solution is a generalization of conventional general solution, uniquely exists and enjoys Lipschitz continuity. As a result, finding numerical general solutions becomes a well-posed problem and solvable in numerical computation. Assuming the data error is sufficiently small, we shall also demonstrate that the exact general solution can be accurately approximated by the numerical general solution and the accuracy is the same order of the data precision. In the applications where only a single solution is needed, every backward accurate solution approximates one of the infinitely many exact solutions and the error occurs in the matrix kernel. We shall also present strategies of computing the numerical general solutions and numerical results.